DOI:
10.1109/SMC.2016.7844460

Document Version
Peer reviewed version

Citation for published version (APA):
Tsai, S. H., Chen, Y-A., Chen, Y-W., Lo, J-C., & Lam, H-K. (2017). A novel stabilization condition for T-S polynomial fuzzy system with time-delay: A sum-of-squares approach. In *2016 IEEE International Conference on Systems, Man, and Cybernetics, SMC 2016 - Conference Proceedings* (pp. 1555-1560). [7844460] Institute of Electrical and Electronics Engineers Inc.. https://doi.org/10.1109/SMC.2016.7844460

A Novel Stabilization Condition for T-S Polynomial Fuzzy System with Time-delay: A Sum-of-Squares Approach

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***Abstract*—A novel stabilization problem for T-S polynomial fuzzy system with time-delay is investigated in this paper. Firstly, a polynomial fuzzy controller for T-S polynomial fuzzy system with time-delay is proposed. In addition, based on polynomial Lyapunov–Krasovskii function and the developed polynomial slack variable matrices, a novel stabilization condition for T-S polynomial fuzzy system with time-delay is presented in terms of sum-of-square (SOS) form. Lastly, nonlinear system with time-delay and a well-known T-S fuzzy system with time-delay are illustrated to demonstrate the feasibility and effectiveness of the proposed results.**

I. Introduction

During the past few decades, Takagi-Sugeno (T-S) fuzzy-model-based control method has aroused immense attention in the study and application of engineering and physical problems. The main reason is that T-S fuzzy-model-based control method provides a systematic design procedure for nonlinear systems. By examination of the modeling problem, T-S fuzzy model can approximate any smooth nonlinear function to any degree of accuracy in any convex compact region [1], [2]. For controller design, the fuzzy controller can be designed through the parallel distributed compensation (PDC) scheme [1], [3] to stabilize the T-S fuzzy system. The output of the overall fuzzy controller is a fuzzy blending of each individual linear controller. Furthermore, linear matrix inequality (LMI) technique has been adopted to address the stability and stabilization problems of T-S fuzzy systems. Therefore, T-S fuzzy-model-based control method has been widely applied to investigate complex nonlinear systems [4]–[6].

Despite the significant successes of the T-S fuzzy-model-based control scheme, there still exist some spaces to further improve some existed results. Therefore, T-S fuzzy model has been extended to T-S polynomial fuzzy model presented in recent years. Each T-S fuzzy model can be expressed as a polynomial model to represent the nonlinear model. Different from the existing LMI approach, sum-of-squares (SOS) technique is adopted to explore the stability and stabilization problem of T-S polynomial fuzzy system. The SOS-based stability/stabilization conditions were thus derived in many studies [7], [8] to ensure the stability of T-S fuzzy polynomial system and facilitate the controller synthesis. In addition, some SOS-based relaxed stability/stabilization conditions are proposed in [9]–[11].

On another research field, it is known that time-delay phenomenon is frequently encountered in many dynamical systems. Recently, in view of possible time-delays in practical systems, analysis and control of fuzzy time-delay systems in T-S fuzzy model have been also addressed in literature [12]–[14]. Therefore, the stability problem for system with time-delay has received much attention and various analysis methods have thus been proposed. For example, in [15], an augmented Lyapunov-Krasovskii function with a triple integral and some augmented vectors was employed to explore the stability problem of T-S fuzzy systems with time-delay. [16] developed a novel augmented Lyapunov function and a delay-dependent stability condition is proposed. In addition, to further reduce the conservativeness of the results, the various constructions of parameter-dependent Lyapunov-Krasovskii functional methods are developed in many studies [15], [17]–[19]. Nevertheless, few studies explore the stability/stabilization problem for T-S polynomial fuzzy system with time-delay. Therefore, based on SOS technique, the stabilization condition for T-S polynomial fuzzy system with time-delay is investigated in this paper.

Motivated by above discussion, this study explores the stabilization problem of T-S polynomial fuzzy time-delay systems by SOS technique. The main contributions of this paper includes: i) a polynomial fuzzy controller for T-S fuzzy system with time-delay is proposed; and ii) a stabilization condition for T-S polynomial fuzzy time-delay systems is proposed in SOS form. The rest of this paper is organized as follows. In Section II, a polynomial fuzzy controller for T-S polynomial fuzzy system with time-delay is proposed. In Section III, based on some slack variable matrices and variable transformations, the delay-dependent stabilization condition for T-S polynomial fuzzy system with time-delay is proposed. In Section IV, a well-known numerical example and a nonlinear model are given to demonstrate that the proposed stabilization condition provides longer allowable delay times than existing ones. Lastly, conclusions are given in Section V.

Notations : $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ represent the set of real n-vector, $m \times n$ matrices. The notation $P > 0(\geq 0)$ means that matrix P is positive (semi) definite. I_n denotes an identity matrix with dimension n, and $0_{m\times n}$ denotes an $m \times n$ dimension zero matrix. The symbol $\star$ indicates transposed elements in SOS, which can be obtained via transpose operation, denoted by ^{T}. $Diag$ denotes a block diagonal matrix.

II. Preliminaries

Consider the following i^{th} rule of the T-S polynomial fuzzy model with time-delay.

$$\textbf{Rule } i: \textbf{IF } s_1(t) \text{ is } M_{i1}(t) \text{ and} \cdots \text{and } s_g(t) \text{ is } M_{ig}(t)$$
$$\textbf{THEN}: \; \dot{x}(t) = A_i(x(t))x(t) + A_{di}(x(t))x(t-d(t)) + B_i(x(t))u(t) \tag{1}$$

where $s_1(t)$, $s_2(t)$, $\cdots$, $s_g(t)$ are premise variables, $M_{ij}(t)$ is fuzzy set, $i,\ j = 1,\ 2, \cdots,\ r$, where r is the number of fuzzy rule. $A_i(x(t)) \in \mathbb{R}^{n\times n}$, $A_{di}(x(t)) \in \mathbb{R}^{n\times n}$ and $B_i(x(t)) \in \mathbb{R}^{n\times 1}$ are polynomial matrices in $x(t)$. $x(t) = \phi(t), t \in [-d_M, 0]$, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t)$ is the control input, $\phi(t)$ is the given initial value, and d_M is the upper bound of delay time. $d(t)$ is the time-varying delay in the state and satisfies that

$$0 \le d(t) \le d_M,\ 0 \le \dot{d}(t) \le d_D. \tag{2}$$

By using a center average defuzzifier, product inference, and a singleton fuzzifier, the overall T-S polynomial fuzzy system with time-delay can be expressed as

$$\begin{aligned}\dot{x}(t) &= \sum_{i=1}^{r} \mu_i(s(t))(A_i(x(t))x(t) + A_{di}x(x(t))x(t-d(t)) \\ &\quad + B_i(x(t))u(t))/\sum_{i=1}^{r}\mu_i(s(t)) \\ &= \sum_{i=1}^{r} h_i(s(t))(A_i(x(t))x(t) + A_{di}(x(t))x(t-d(t)) \\ &\quad + B_i(x(t))u(t))\end{aligned} \tag{3}$$

where, $\mu_i(s(t)) = \prod_{i=1}^{p} M_{ij}(s(t))$, $h_i = \mu_i(s(t))/\sum_{i=1}^{r}\mu_i(s(t))$. Two basic properties of $\mu_i(s(t))$ are $\mu_i(s(t)) \ge 0$ and $\sum_{i=1}^{r}\mu_i(s(t)) > 0$. It is clear that $h_i(s(t)) \ge 0$, and $\sum_{i=1}^{r} h_i(s(t)) = 1$.

A. Polynomial Fuzzy Controller Design

Based on PDC scheme, the polynomial fuzzy controller for the polynomial fuzzy system with time-delay (1) is designed as follow.

$$\textbf{Controller Rule } i:$$
$$\textbf{IF } s_1(t) \text{ is } M_{i1}(t) \text{ and } \cdots \text{ and } s_g(t) \text{ is } M_{ig}(t)$$
$$\textbf{THEN } u(t) = K_i(x)x(t) \tag{4}$$

Substituting (4), the closed-loop polynomial fuzzy system (3) can be represented as:

$$\begin{aligned}\dot{x}(t) &= \sum_{i=1}^{r} h_i(s(t)) \sum_{i=j}^{r} h_j(s(t))((A_i(x) + B_i(x)K_j(x))x(t) \\ &\quad + A_{di}(x)x(t-d(t))).\end{aligned} \tag{5}$$

In next section, the stabilization problem for (3) will be discussed.

III. Main Results

In this section, the stabilization condition for T-S polynomial fuzzy system with time-delay will be explored. The main result is proposed in the following theorem.

Theorem 1: For T-S polynomial fuzzy time-delay system (3), there exists a polynomial fuzzy controller (4) such that closed-loop T-S polynomial fuzzy time-delay system (5) is asymptotically stable if there exists matrices P, symmetric polynomial matrices $\bar{E}(x)$, $\bar{W}(x)$, $\bar{Z}(x) \in \mathbb{R}^{n\times n}$, polynomial matrix $\bar{S}(x) \in \mathbb{R}^{n\times n}$, scalar $\bar{\sigma} > 0$, and $0 \le d(t) \le d_M$, $0 \le \dot{d}(t) \le d_D$ satisfying the following SOS conditions, where $i, j = 1, ..., r$, $i < j \le r$ are nonnegative polynomials for all x.

$$\begin{aligned}
&\bar{v}_1^T \left(P - \bar{\epsilon}_1(x)I\right)\bar{v}_1 && \text{is SOS}\\
&\bar{v}_2^T \left(\bar{E}(x) - \bar{\epsilon}_2(x)I\right)\bar{v}_2 && \text{is SOS}\\
&\bar{v}_3^T \left(\bar{W}(x) - \bar{\epsilon}_3(x)I\right)\bar{v}_3 && \text{is SOS}\\
&\bar{v}_4^T \left(\bar{Z}(x) - \bar{\epsilon}_4(x)I\right)\bar{v}_4 && \text{is SOS}\\
&\bar{v}_5^T \left(\bar{W}(x) - \bar{Z}(x) - \bar{\epsilon}_5(x)I\right)\bar{v}_5 && \text{is SOS}\\
-&\bar{v}_6^T \left(\bar{\Lambda}_{ii}(x) + \bar{\epsilon}_{6ii}(x)I\right)\bar{v}_6 && \text{are SOS}\\
-&\bar{v}_6^T \left(\bar{\Lambda}_{ij}(x) + \bar{\Lambda}_{ji}(x) + \bar{\epsilon}_{6ij}(x)I\right)\bar{v}_6 && \text{are SOS}
\end{aligned}$$

where

$$\bar{\Lambda}_{ij}(x) = \begin{bmatrix} \bar{\Lambda}_{ij}(1,1) & \bar{\Lambda}_{ij}(1,2) & \bar{\Lambda}_{ij}(1,3) & 0_{N\times N} \\ \star & \bar{\Lambda}_{ij}(2,2) & \bar{\Lambda}_{ij}(2,3) & \bar{\Lambda}_{ij}(2,4) \\ \star & \star & \bar{\Lambda}_{ij}(3,3) & 0_{N\times N} \\ \star & \star & \star & \bar{\Lambda}_{ij}(4,4) \end{bmatrix} \le 0$$

$$\begin{aligned}
\bar{\Lambda}_{ij}(1,1) &= (A_i(x)P + \star) + (B_i(x)\bar{K}_j(x) + \star) + \bar{E}(x)\\
\bar{\Lambda}_{ij}(1,2) &= A_{di}(x)P + \bar{S}^T(x)\\
\bar{\Lambda}_{ij}(1,3) &= \sigma P A_i^T(x) + \sigma \bar{K}_j(x)B_i^T(x)\\
\bar{\Lambda}_{ij}(2,2) &= -(1-d_D)\bar{E}(x) - (\bar{S}(x) + \star)\\
\bar{\Lambda}_{ij}(2,3) &= \sigma P A_{di}^T(x),\ \bar{\Upsilon}_{3ii}(2,4) = \bar{S}(x)\\
\bar{\Lambda}_{ij}(3,3) &= d_M\bar{W}(x) - 2\sigma P\\
\bar{\Lambda}_{ij}(4,4) &= -d_M^{-1}\bar{Z}
\end{aligned}$$

$\bar{v}_i \in \mathbb{R}^n$, $i = 1, ..., 6$ denote the vectors that are independent of $x(t)$ and $x(t-d(t))$.

Proof: Choose the following Lyapunov-Krasovskii functional candidate

$$V(x) = V_1(x) + V_2(x) + V_3(x) \tag{6}$$

where

$$\begin{aligned}
V_1(x) &= x^T(t)P^{-1}x(t)\\
V_2(x) &= \int_{t-d(t)}^{t} x^T(\alpha)Ex^T(\alpha)d\alpha\\
V_3(x) &= \int_{0}^{-d_M}\int_{t+v}^{t} \dot{x}^T(\alpha)W\dot{x}(\alpha)d\alpha dv.
\end{aligned}$$

E and W are positive symmetric matrices.

By Newton-Leibniz formula, the time derivative of (6) becomes:

$$\dot{V}(x) = \dot{V}_1(x) + \dot{V}_2(x) + \dot{V}_3(x) \tag{7}$$

where

$$\begin{aligned}\dot{V}_1(x) &= 2\dot{x}^T(t)P^{-1}x(t)\\ &= 2\sum_{i=1}^{r} h_i(s(t))\sum_{j=1}^{r} h_j(s(t))((A_i(x)\\ &\quad + B_i(x)K_j(x))x(t) + A_{di}(x)x(t-d(t)))^T P^{-1}x(t)\\ &= \sum_{i=1}^{r} h_i(s(t))\sum_{j=1}^{r} h_j(s(t))(\hat{x}^T(t)((A_i^T(x)P^{-1}+\star)\\ &\quad + (K_j^T(x)B_i^T(x)P^{-1}+\star))\hat{x}(t)\\ &\quad + (\hat{x}^T(t-d(t))(A_{di}^T(x)P^{-1})x(t)+\star)) \qquad (8)\\ \dot{V}_2(x) &= x^T(t)Ex(t)\\ &\quad - (1-\dot{d}(t))x^T(t-d(t))Ex(t-d(t)) \qquad (9)\\ \dot{V}_3(x) &= d_M\dot{x}^T(t)W\dot{x}(t)\\ &\quad - \int_{t-d_M}^{t}\dot{x}^T(x(\alpha))W\dot{x}(x(\alpha))d\alpha. \qquad (10)\end{aligned}$$

From (9) and $0 \le \dot{d}(t) \le d_D$, we have

$$\begin{aligned}\dot{V}_2(x) &\le x^T(t)Ex(t)\\ &\quad - (1-d_D)x^T(t-d(t))Ex(t-d(t)). \qquad (11)\end{aligned}$$

Similarly, from (10) and $0 \le d(t) \le d_M$, yield

$$\begin{aligned}\dot{V}_3(x) &\le d_M\dot{x}^T(t)W\dot{x}(t)\\ &\quad - \int_{t-d(t)}^{T}\dot{x}^T(x(\alpha))W\dot{x}(x(\alpha))d\alpha. \qquad (12)\end{aligned}$$

Furthermore, the following matrix equations are held

$$\begin{aligned}\Pi_5 =& 2x^T(t-d(t))S(x)(x(t)-x(t-d(t))\\ &- \int_{t-d(t}^{T}\dot{x}(\alpha)d\alpha) = 0 \qquad (13)\\ \Pi_6 =& d_M x^T(t-d(t))S(x)W^{-1}S^T(x)x(t-d(t))\\ &- \int_{t-d_M}^{t} x^T(t-d(t))S(x)W^{-1}S^T(x)\\ &\times x(t-d(t))d\alpha = 0 \qquad (14)\\ \Pi_7 =& \sum_{i=1}^{r} h_i(s(t))\sum_{j=1}^{r} h_j(s(t))2(\dot{x}^T(t)\sigma P^{-1}((A_i(x)\\ &+ B_i(x)K_j(x))x(t)\\ &+ A_{di}(x)x(t-d(t)) - \dot{x}(t))) = 0. \qquad (15)\end{aligned}$$

According to (14) and $0 \le d(t) \le d_M$, we can obtain

$$\begin{aligned}\Pi_5 \le& d_M x^T(t-d(t))S(x)W^{-1}S^T(x)x(t-d(t))\\ &- \int_{t-d(t)}^{t} x^T(t-d(t))S(x)W^{-1}S^T(x)\\ &\times x(t-d(t))d\alpha = 0 \qquad (16)\end{aligned}$$

Concluding by above results with $Z < W$, we can get

$$\begin{aligned}\dot{V}(x) \le& \sum_{i=1}^{r} h_i(s(t))\sum_{j=1}^{r} h_j(s(t))(\Upsilon_{ij}(x)\\ &+ d_M x^T(t-d(t))S(x)W^{-1}S(x)x(t-d(t))\\ &- \int_{t-d(t)}^{t}((\dot{x}^T(t)W + x^T(t-d(t))S(x))W^{-1}\\ &\times (W\dot{x}(t) + S^T(x)x(t-d(t))))ds)\\ \le& \sum_{i=1}^{r} h_i(s(t))\sum_{j=1}^{r} h_j(s(t))(\Upsilon_{1ij}(x)\\ &+ d_M x^T(t-d(t))S(x)Z^{-1}S^T(x)x(t-d(t)))\\ =& \sum_{i=1}^{r} h_i^2(s(t))(\Upsilon_{1ii}(x) + d_M x^T(t-d(t))S(x)Z^{-1}\\ &\times S^T(x)x(t-d(t))) + \sum_{i<j}^{r} h_i(s(t))h_j(s(t))(\Upsilon_{1ij}(x)\\ &+ \Upsilon_{1ji}(x) + d_M x^T(t-d(t))S(x)Z^{-1}\\ &\times S^T(x)x(t-d(t))) \qquad (17)\end{aligned}$$

where

$$\begin{aligned}\Upsilon_{1ij}(x) =& d_M\dot{x}^T(t)W\dot{x}(t) + x^T(t)Ex(t) - (1-d_D)\\ &\times x^T(t-d(t))Ex(t-d(t)) + x^T(t)(P^{-1}(A_i(x)\\ &+ B_i(x)K_j(x)) + \star)x(t) + (x^T(t)P^{-1}A_{di}(x)\\ &\times x(t-d(t)) + \star) + 2x^T(t-d(t))S(x)(x(t)\\ &- x(t-d(t))) + 2\sigma\dot{x}(t)P^{-1}(A_i(x)x(t)\\ &+ A_{di}(x)x(t-d(t)) - \dot{x}(t))\\ =& \eta^T(t)\Upsilon_{2ij}(x)\eta(t)\end{aligned}$$

$$\begin{aligned}\Upsilon_{2ij}(x) &= \begin{bmatrix} \Upsilon_{2ij}(1,1) & \Upsilon_{2ij}(1,2) & \Upsilon_{2ij}(1,3)\\ \star & \Upsilon_{2ij}(2,2) & \Upsilon_{2ij}(2,3)\\ \star & \star & \Upsilon_{2ij}(3,3)\end{bmatrix}\\ \Upsilon_{2ij}(1,1) &= (P^{-1}A_i(x)+\star) + (P^{-1}B_i(x)K_j(x)+\star) + E\\ \Upsilon_{2ij}(1,2) &= (P^{-1}A_{di}(x) + S^T(x)\\ \Upsilon_{2ij}(1,3) &= \sigma A_i^T(x)P^{-1} + \sigma K_j^T(x)B_i(x)P^{-1}\\ \Upsilon_{2ij}(2,2) &= -(1-\sigma_D)E - (S(x)+\star)\\ \Upsilon_{2ij}(2,3) &= \sigma A_{di}^T(x)P^{-1}\\ \Upsilon_{2ij}(3,3) &= d_M W - 2\sigma P^{-1}\\ \eta^T(t) &= \begin{bmatrix} x^T(t) & x^T(t-d(t)) & \dot{x}^T(t)\end{bmatrix}^T\end{aligned}$$

In order for $\dot{V}(x) \le 0$, the following conditions should be satisfied

$$\begin{aligned}&\eta^T(t)\Upsilon_{2ii}(x)\eta(t) + d_M x^T(t-d(t))S(x)Z^{-1}\\ &\quad\times S^T(x)x(t-d(t)) \le 0 \qquad (18)\\ &\eta^T(t)(\Upsilon_{2ij}(x) + \Upsilon_{2ji}(x))\eta(t) + d_M x^T(t-d(t))S(x)\\ &\quad\times Z^{-1}S^T(x)x(t-d(t)) \le 0. \qquad (19)\end{aligned}$$

Firstly, let us consider (18). Applying Schur complement formula to (18), we can obtain

$$\Upsilon_{3ii}(x) = \begin{bmatrix} \Upsilon_{2ii}(1,2) & \Upsilon_{2ii}(1,2) & \Upsilon_{2ii}(1,3) & 0_{N\times N} \\ \star & \Upsilon_{2ii}(2,2) & \Upsilon_{2ii}(2,3) & S(x) \\ \star & \star & \Upsilon_{2ii}(3,3) & 0_{N\times N} \\ \star & \star & \star & -d_M^{-1}Z \end{bmatrix} \leq 0. \quad (20)$$

Pre- and post-multiplying both sides of (20) with $Diag[P\ P\ P\ P]$ and its transpose, and defining $PE(x)P = \bar{E}(x)$, $PWP = \bar{W}(x)$, $PZP = \bar{Z}$, $PS(x)P = \bar{S}(x)$ and $K_j(x)P = \bar{K}_j(x)$, yield

$$\bar{\Lambda}_{ii}(x) = \begin{bmatrix} \bar{\Lambda}_{ii}(1,1) & \bar{\Lambda}_{ii}(1,2) & \bar{\Lambda}_{ii}(1,3) & 0_{N\times N} \\ \star & \bar{\Lambda}_{ii}(2,2) & \bar{\Lambda}_{ii}(2,3) & \bar{\Lambda}_{ii}(2,4) \\ \star & \star & \bar{\Lambda}_{ii}(3,3) & 0_{N\times N} \\ \star & \star & \star & \bar{\Lambda}_{ii}(4,4) \end{bmatrix} \leq 0$$

where

$$\begin{aligned} \bar{\Lambda}_{ii}(1,1) =&(A_i(x)P + \star) + (B_i(x)\bar{K}_i(x) + \star) + \bar{E}(x) \\ \bar{\Lambda}_{ii}(1,2) =&A_{di}(x)P + \bar{S}^T(x) \\ \bar{\Lambda}_{ii}(1,3) =&\sigma P A_i^T(x) + \sigma \bar{K}_i(x) B_i^T(x) \\ \bar{\Lambda}_{ii}(2,2) =&-(1-d_D)\bar{E}(x) - (\bar{S}(x) + \star) \\ \bar{\Lambda}_{ii}(2,3) =&\sigma P A_{di}^T(x),\ \bar{\Upsilon}_{3ii}(2,4) = \bar{S}(x) \\ \bar{\Lambda}_{ii}(3,3) =&d_M \bar{W}(x) - 2\sigma P \\ \bar{\Lambda}_{ii}(4,4) =&-d_M^{-1}\bar{Z} \end{aligned}$$

Through a similar procedure, we can obtain $\bar{\Lambda}_{ij}(x) + \bar{\Lambda}_{ji}(x) \leq 0$ from (19). Therefore, if SOS conditions in Theorem 1 are satisfied, then $\dot{V}(x) \leq 0$. In addition, it is easy to see that if SOS conditions in Theorem 1 are satisfied for $x \neq 0$ then $\bar{\Lambda}_{ii}(x) < 0$ and $\bar{\Lambda}_{ij}(x) + \bar{\Lambda}_{ji}(x) < 0$. Thus, $\dot{V}(x) < 0$ for all $x \neq 0$. Hence, the closed-loop polynomial fuzzy time-delay system (5) is asymptotically stable. Moreover, if the SOS conditions are satisfied for $x \neq 0$, then $\dot{V}(x) < 0$. This completes the proof. ■

IV. SIMULATION

In this section, a numerical example and a nonlinear model are provided to illustrate the effectiveness of the previously developed methods.

Example 1:

Consider the following T-S fuzzy time-delay system given in [18], [20]–[26].

Rule 1 : **IF** $x_1(t)$ is $M_1(x(t))$
THEN $\dot{x}(t) = A_1x(t) + A_{d1}x(t-d(t)) + B_1u(t)$
Rule 2 : **IF** $x_1(t)$ is $M_2(x(t))$
THEN $\dot{x}(t) = A_2x(t) + A_{d2}x(t-d(t)) + B_2u(t)$

where

$$A_1 = \begin{bmatrix} 0 & 0.6 \\ 0 & 1 \end{bmatrix},\ A_{d1} = \begin{bmatrix} 0.5 & 0.9 \\ 0 & 2 \end{bmatrix},\ B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix},\ A_{d2} = \begin{bmatrix} 0.9 & 0 \\ 1 & 1.6 \end{bmatrix},\ B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

TABLE I. COMPARISONS AMONG VARIOUS METHODS

Method	Maximum allowed d_2
Theorem 2 of [20]	0.152
Theorem 1 of [21]	0.230
Corollary 2 of [22]	0.257
Theorem 1 of [22]	0.266
Theorem 1 of [23]	0.491
Theorem 3 of [18]	0.842
Theorem 1 of [18]	0.900
Theorem 2 of [24]	1.050
Corollary 2 of [25]	1.095
Theorem 1 of [26]	1.380
Theorem 2 of [27]	1.432
Theorem 2 of [28]	1.482
Theorem 1 of this paper	**1.5003**

The membership functions are defined as: $M_1(x(t)) = 1/(1+\exp(-2x_1(t)))$, $M_2(x(t)) = 1 - M_1(x(t))$. By Theorem 1 with $\sigma = 0.1$, we can obtain the maximum allowable delay time 1.5003 sec. with P and controllers $\bar{K}_1$ and $\bar{K}_2$ where $K_1 = \bar{K}_1 P^{-1}$ and $K_2 = \bar{K}_2 P^{-1}$.

Table I shows a comparison of the maximal allowable delay times obtained in ten studies. It can be seen from Table I that the proposed stabilization conditions can provide the largest allowable delay time. The state responses for T-S fuzzy time-delay system with $x(0) = [2 \quad 1]^T$ is shown in Fig. 1. Simulation results show that the trajectories of fuzzy time-delay system converge to the equilibrium state after some transient times.

Example 2:

Consider the following nonlinear system.

$$\begin{aligned} \dot{x}_1(t) &= -x_1(t) + x_1^2(t) + x_1^2(t)x_2(t) + x_2(t) + x_1(t)u(t) \\ \dot{x}_2(t) &= -\sin x_1(t) - x_2(t) + u(t) \end{aligned}$$

To illustrate the proposed results, we assume that the system $x_2(t)$ is perturbed by time-delay and the delayed model is given as

$$\begin{aligned} \dot{x}_1(t) =& -x_1(t) + x_1^2(t) + ax_1^2(t)x_2(t) + ax_2(t) + x_1(t)u(t) \\ &+ (1-a)x_1^2(t)x_2(t-d(t)) + (1-a)x_2(t-d(t)) \\ \dot{x}_2(t) =& -\sin x_1(t) - ax_2(t) - (1-a)x_2(t-d(t)) + u(t) \end{aligned} \quad (21)$$

The constant a is the retarded coefficient, which satisfies the conditions $a \in [0, 1]$. The limits 1 and 0 correspond to no delay term and to a complete delay term, respectively. In this example, we assume $a = 0.7$. By the fuzzy modeling method in [7], (21) can be expressed as the following T-S polynomial fuzzy model with time-delay.

Rule 1 : **IF** $x_1(t)$ is 0
THEN $\dot{x}(t) = A_1(x)x(t) + A_{d1}(x)x(t-d(t)) + B_1(x)u(t)$
Rule 2 : **IF** $x_1(t)$ is $-\pi$ or π
THEN $\dot{x}(t) = A_2(x)x(t) + A_{d2}(x)x(t-d(t)) + B_2(x)u(t)$

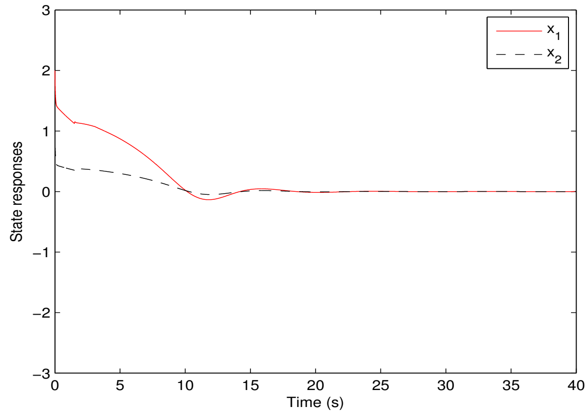

Fig. 1. State response for T-S fuzzy system with time-delay

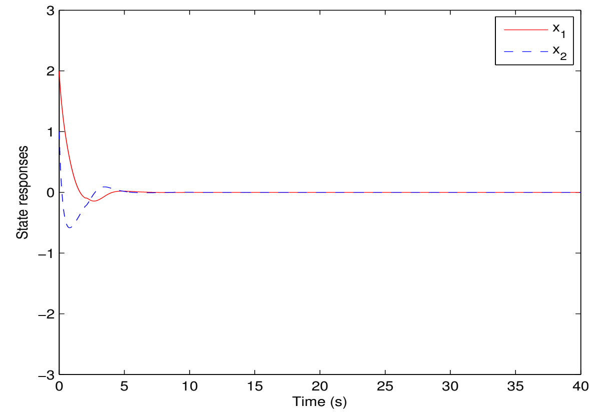

Fig. 2. State responses of Example 2.

where

$$A_1(x) = \begin{bmatrix} -1 + x_1(t) & ax_1^2(t) + a \\ -1 & -a \end{bmatrix}, \; B_1(x) = \begin{bmatrix} x_1(t) \\ 1 \end{bmatrix},$$

$$A_2(x) = \begin{bmatrix} -1 + x_1(t) & ax_1^2(t) + a \\ 0.2172 & -a \end{bmatrix}, \; B_2(x) = \begin{bmatrix} x_1(t) \\ 1 \end{bmatrix},$$

$$A_{d1}(x) = A_{d2}(x) = \begin{bmatrix} 0 & (1-a)x_1^2 + (1-a) \\ 0 & -(1-a) \end{bmatrix}.$$

The delay time $d(t) = 1 + \sin(t)$. The membership functions are defined as: $M_1(x(t)) = (\sin(x_1(t)) + 0.2172x_1(t))/(1.2172x_1(t))$, $M_2(x(t)) = 1 - M_1(x(t))$. From $d(t)$, it can be seen that $d_M = 2$ and $d_D = 1$. By utilizing the MATLAB SOS toolbox to solve the convex optimization problem in Theorem 1 with $\sigma = 1$, the polynomial fuzzy controller are obtained as follows

$$P = \begin{bmatrix} 1.797 \times 10^{-8} & 8.645 \times 10^{-10} \\ 8.645 \times 10^{-10} & 2.912 \times 10^{-8} \end{bmatrix}$$
$$K_1(x) = [\; -0.8049x_1 \quad -0.5800x_1 \;]$$
$$K_2(x) = [\; -1.4259x_1 \quad -0.5641x_1 \;].$$

Fig. 2 shows the state responses of (21) under initial values $[2 \quad 1]^T$, and it can be seen that the trajectories of the trajectories converge to equilibrium state after few seconds.

By utilizing the polynomial fuzzy controller, the state responses for the closed-loop nonlinear system (21) with initial values $[2\ 1]^T$, $[1\ 2]^T$, $[2\ -1]^T$, $[1\ -2]^T$, $[-2\ -1]^T$, $[-1\ -2]^T$, $[-1\ 2]^T$ and $[-2\ 1]^T$ are shown in Fig. 3.

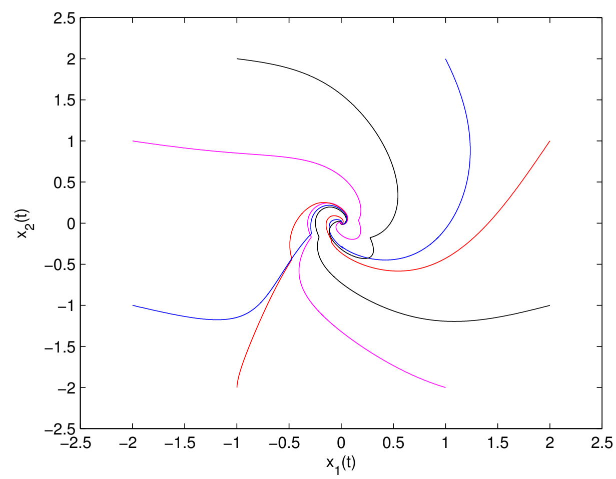

Fig. 3. Behaviors in $x_1 - x_2$ plane (with feedback).

V. Conclusion

In this paper, a stabilization problem for T-S polynomial fuzzy system with time-delay is investigated. By utilizaing PDC concept, a polynomial fuzzy controller for T-S polynomial fuzzy system with time-delay is proposed. In addition, based on some slack variable matrices, a delay-dependent stabilization condition for T-S polynomial fuzzy system with time-delay is proposed. Furthermore, the results can be formulated in terms of SOS forms. A nonlinear system with time-delay and a well-known numerical example are given to illustrate the effectiveness of the proposed methods.

Acknowledgment

This work was supported by the National Science Council of Taiwan, R.O.C., under Grant MOST 104-2221-E-027-066-.

References

[1] K. Tanaka and H. O. Wang, *Fuzzy Control Systems Design and Analysis: A Linear Matrix Inequality Approach.* New York. Wiley, 2001.

[2] K. Zeng, N. Y. Zhang, and W. Xu, "A comparative study on sufficient conditions for Takagi-Sugeno fuzzy systems as universal approximators," *Fuzzy Set. Syst.*, vol. 8, no. 6, pp. 773–780, Dec. 2000.

[3] L. Wu, X. Yang, and H. K. Lam, "Dissipativity analysis and synthesis for discrete-time T-S fuzzy stochastic systems with time-varying delay," *IEEE Trans. Fuzzy Syst.*, vol. 22, no. 2, pp. 380–394, Apr. 2014.

[4] M. H. Asemani and V. J. Majd, "A robust H_∞ non-PDC design scheme for singularly perturbed T-S fuzzy systems with immeasurable state variables," *IEEE Trans. Fuzzy Syst.*, vol. 23, no. 3, pp. 525–541, Jun. 2015.

[5] S. H. Tsai, C. Sun, J. C. Lo, and H. K. Lam, "Relaxed stabilization of T-S fuzzy systems with time-delay," in *2013 IEEE International Conference on Fuzzy Systems(FUZZ)*, Hyderabad, Jul. 2013, pp. 1–6.

[6] S.-H. Tsai, "Robust H_∞ stabilization conditions for a class of uncertain ts fuzzy neutral systems with disturbance," *Neurocomputing*, vol. 193, no. 12, pp. 68–80, Jun. 2016.

[7] K. Tanaka, H. Yoshida, H. Ohtake, and H. Wang, "A sum-of-squares approach to modeling and control of nonlinear dynamical systems with polynomial fuzzy systems," *IEEE Trans. Fuzzy Syst.*, vol. 17, no. 4, pp. 911–922, Aug. 2009.

[8] K. Tanaka, H. Ohtake, T. Seo, M. Tanaka, and H. Wang, "Polynomial fuzzy observer designs: A sum-of-squares approach," *IEEE Trans. Syst. Man Cybern. Part B-Cybern.*, vol. 42, pp. 1330–1342, Apr. 2012.

[9] J. C. Lo, Y. T. Lin, W. S. Chang, and F. Y. Lin, "SOS-based fuzzy stability analysis via homogeneous Lyapunov functions," in *2014 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE)*, Beijing, Jun. 2014, pp. 2300–2305.

[10] S. Chae and S. Nguang, "SOS based robust H_∞ fuzzy dynamic output feedback control of nonlinear networked control systems," *IEEE Trans. Cybernetics*, pp. 1204–1213, Jul. 2014.

[11] Y. Chen, M. Tanaka, K. Tanaka, H. Ohtake, and H. Wang, "Discrete polynomial fuzzy systems control," *IET Control Theory and Applications*, vol. 8, no. 4, pp. 288–296, Sep. 2013.

[12] W. Yang and S. Tong, "Output feedback robust stabilization of switched fuzzy systems with time-delay and actuator saturation," *Neurocomputing*, vol. 164, no. 1, pp. 173–181, Sep. 2015.

[13] S.-H. Tsai and C.-J. Fang, "A novel relaxed stabilization condition for a class of T-S time-delay fuzzy systems," in *2014 IEEE International Conference on Fuzzy Systems(FUZZ)*, Beijing, Jul. 2014, pp. 2294–2299.

[14] S.-H. Tsai, "Delay-dependent robust stabilisation for a class of fuzzy bilinear systems with time-varying delays in state and control input," *International Journal of Systems Science*, vol. 45, no. 3, pp. 187–201, Apr. 2014.

[15] O. Kwon, M. Park, S. Lee, and J. Park, "Augmented Lyapunov-Krasovskii functional approaches to robust stability criteria for uncertain Takagi-Sugeno fuzzy systems with time-varying delays," *Fuzzy Sets and Systems*, vol. 201, pp. 1–19, Aug. 2012.

[16] H. Zeng, J. Park, J. Xia, and S. Xiao, "Improved delay-dependent stability criteria for T-S fuzzy systems with time-varying delay," *Applied Mathematics and Computation*, vol. 235, no. 25, pp. 492–501, May. 2014.

[17] Z. Zhang, C. Lin, and B. Chen, "New stability and stabilization conditions for T-S fuzzy systems with time delay," *Fuzzy Sets and Systems*, vol. 263, no. C, pp. 82–91, Mar. 2015.

[18] Y. Zhao, H. Gao, J. Lam, and B. Du, "Stability and stabilization of delayed T-S fuzzy systems: A delay partitioning approach," *IEEE Trans. Fuzzy Syst.*, vol. 17, no. 4, pp. 750–762, Aug. 2009.

[19] S. H. Kim, "Improved approach to robust H_∞ stabilization of discrete-time T-S fuzzy systems with time-varying delays," *IEEE Trans. Fuzzy Syst.*, vol. 18, no. 5, pp. 1008–1015, Oct. 2010.

[20] B. Chen and X. Liu, "Delay-dependent robust H_∞ control for T-S fuzzy systems with time delay," *IEEE Trans. Fuzzy Syst.*, vol. 13, no. 4, pp. 544–556, Aug. 2005.

[21] X. Guan and C. Chen, "Delay-dependent guaranteed cost control for T-S fuzzy systems with time delays," *IEEE Trans. Fuzzy Syst.*, vol. 12, no. 2, pp. 236–49, Apr. 2004.

[22] H. Wu and H. Li, "New approach to delay-dependent stability analysis and stabilization for continuous-time fuzzy systems with time-varying delay," *IEEE Trans. Fuzzy Syst.*, vol. 15, no. 3, pp. 482–493, Jun. 2007.

[23] G. Hamdi, E. H. Ahmed, and C. Mohamed, "Robust control of T-S fuzzy systems with time-varying delay using new approach," *IEEE Trans. Fuzzy Syst.*, vol. 15, no. 3, pp. 4168–4173, Dec. 2009.

[24] L. Li and X. Liu, "New results on delay-dependent robust stability criteria of uncertain fuzzy systems with state and input delays," *Information Sciences*, vol. 179, no. 8, pp. 1134–1148, Mar. 2009.

[25] C. Peng, Y. Tian, and E. Tian, "Improved delay-dependent robust stabilization conditions of uncertain T-S fuzzy systems with time-varying delay," *Fuzzy Set. Syst.*, vol. 159, no. 20, pp. 2731–2729, Oct. 2008.

[26] F. Liu, M. Wu, Y. He, and R. Yokohama, "New delay-dependent stability criteria for T-S fuzzy systems with time-varying delay," *Fuzzy Set. Syst.*, vol. 161, no. 15, pp. 2033–2042, Aug. 2010.

[27] L. Zhao, H. Gao, and H. R. Karimi, "Robust stability and stabilization of uncertain T-S fuzzy systems with time-varying delay: An input-output approach," *IEEE Trans. Fuzzy Syst.*, vol. 21, no. 5, pp. 883–897, Oct. 2013.

[28] S.-H. Tsai, Y.-A. Chen, and J.-C. Lo, "A novel stabilization condition for a class of T-S fuzzy time-delay systems," *Neurocomputing*, vol. 175, no. A, pp. 223–232, Jan. 2016.